# Characterizing Transitioning in Chaotic Models

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## Abstract

In order to understand how complicated physical systems behave, we study idealized systems instead and interpret the qualitative behavior. In order to understand how non-linear, chaotic systems transition into new parameter sets, we characterize the distribution of dynamical points over the manifold of trajectories (also known as the "strange attractor") for the Lorenz model under two regimes. We consider the effects of variation of just one of the three parameters of the Lorenz model. First, we establish measures of shape of the distribution over the manifold for a range of static values of that parameter. Then, the same measures of shape are calculated for the trajectory that results when a parameter is ramped linearly in time. Statistical comparison of these distributions will be used to describe the evolution of the attractor. This simple model can illustrate how such non-linear, chaotic systems behave when the parameters of the system vary.

## Introduction

The Lorenz model is a nonlinear system developed by Edward Lorenz in 1963 to understand rolling convective behavior (rolling motion of heat transport) of fluids such as the atmosphere. In its simplified form, the set of differential equations is:

$$\frac{dx}{dt} = \sigma\left(y - x\right) \quad \frac{dy}{dt} = x\left(\rho - z\right) - y \quad \frac{dz}{dt} = xy - \beta z$$

We choose the Lorenz model because this system is:

- Nonlinear
- Chaotic, for some sets $[\rho, \sigma, \beta]$
- Deterministic
- Easy to compute and implement
- Does not change drastically over the selected range of $\rho$

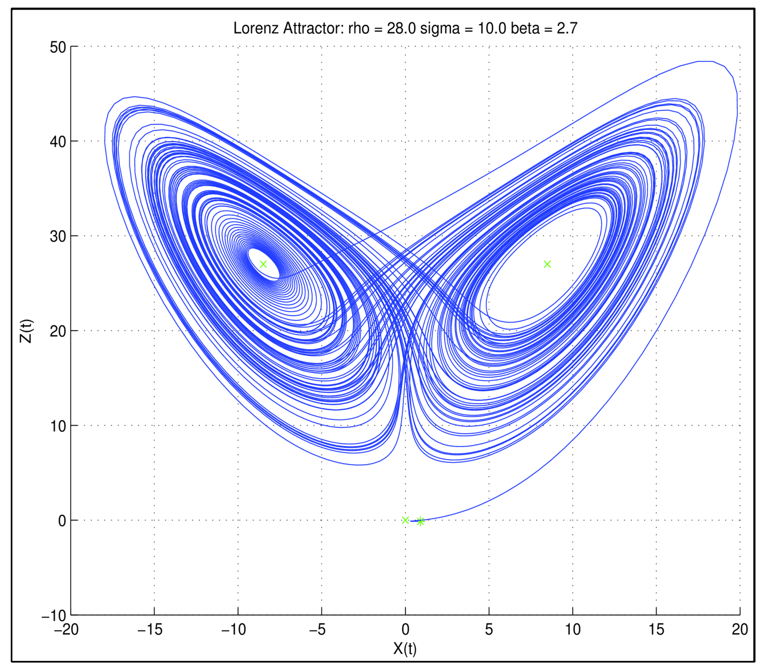


Fig 1: Typical Trajectory, static $\rho$

## Methods

We will implement the Lorenz model numerically. We will compare simulations of the Lorenz model where the parameters are held static in time, with a simulation where the parameters are changed over time.

For the fixed parameter simulations we take parameters which produce chaos in the Lorenz model: $\sigma$ = 10, $\beta$ = 8/3, and $\rho$ chosen from2000 values in the range [28,30].

For the simulation with dynamic parameter values, we take the same $\sigma$ and $\beta$ values, but take $\rho$ according to the piecewise definition:

$$\rho(t) = \begin{cases} 28 & \text{if } t < t_1 \\ 28 - (30 - 28)\left(\frac{t - t_1}{t_2 - t_1}\right) & \text{if } t_1 < t < t_2 \\ 30 & \text{if } t > t_2 \end{cases}$$

## Methods cont'd

Using MATLAB we integrate the set of differential equations in time from 0 to 500 (60k pts) for each fixed value of $\rho$. One such trajectory is pictured in Figure 1. Statistics and measures of shape are then calculated for each trajectory, with each value of static $\rho$.

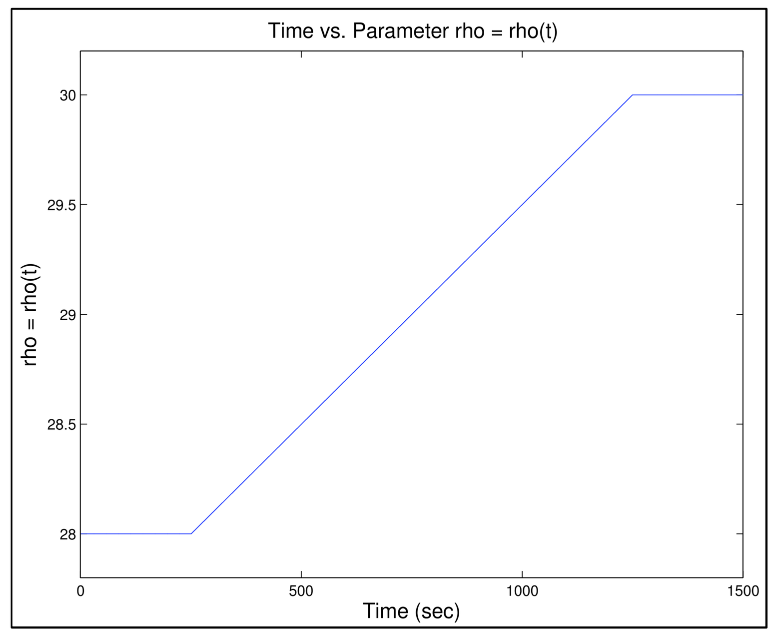


Fig 2: Plot of $\rho = \rho(t)$

Next, a trajectory with dynamic $\rho(t)$ is integrated over time 1500, taking $t_1$ as 250 and $t_2$ as 1250 (see Figure 2). We slice the dynamic $\rho(t)$ trajectory into segments 10k pts long and compute the same statistics and measures of shape.

We compare various measures of shape to determine if there are significant differences between the systems where parameters are constant in time and where parameters are dynamic in time:

- Location of center of mass
- $2^{nd}$ moments (inertial)
- Skewness $S = \frac{E\left(x - \mu\right)^3}{\sigma^3}$
- Kurtosis $K = \frac{E\left(x - \mu\right)^4}{\sigma^4}$

## Results: $\rho$ static

After collecting trajectories integrated in time over 60k pts, measures of shape are drawn from each trajectory. The attractor basin (Figure 1) is well defined and does not change topological shape over the selected range of static $\rho$. We find the location of the height of the center of mass above the foci, the moments of inertia, the skewnesses and the kurtoses of each fixed parameter trajectory and plot versus $\rho$ value. These are presented in Figures 3 through 6 below.

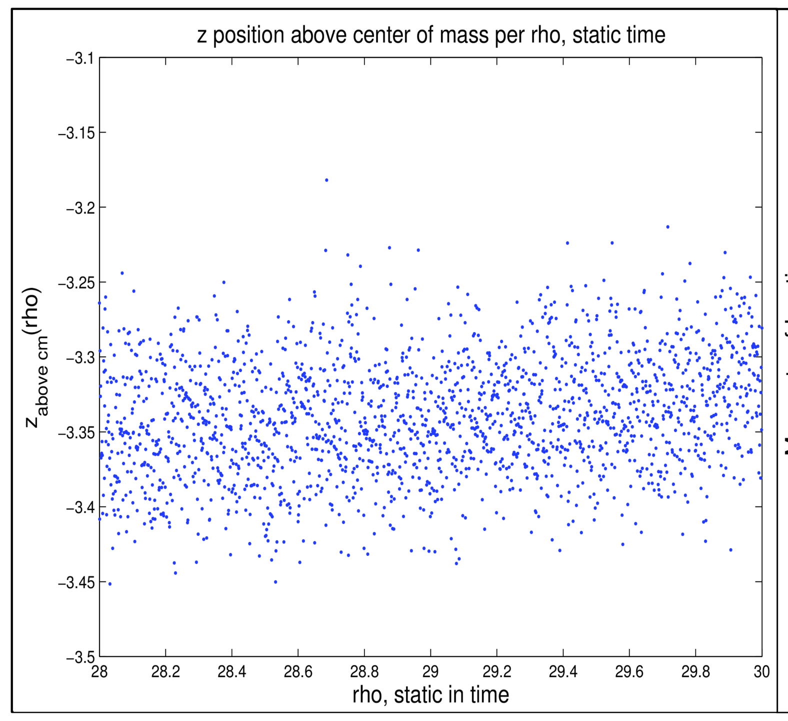


Fig 3: Height of zcm vs static ρ

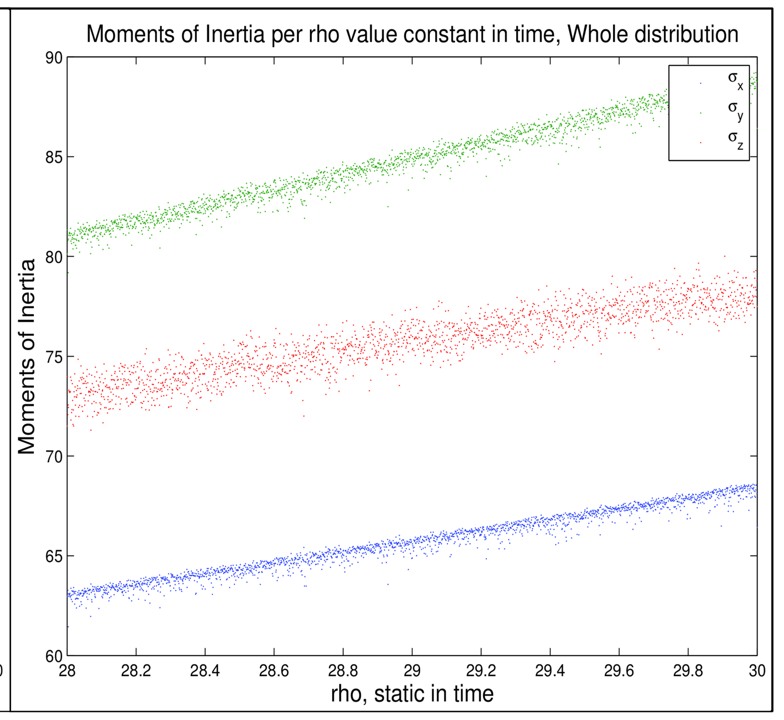


Fig 4: 2nd moment vs. static ρ

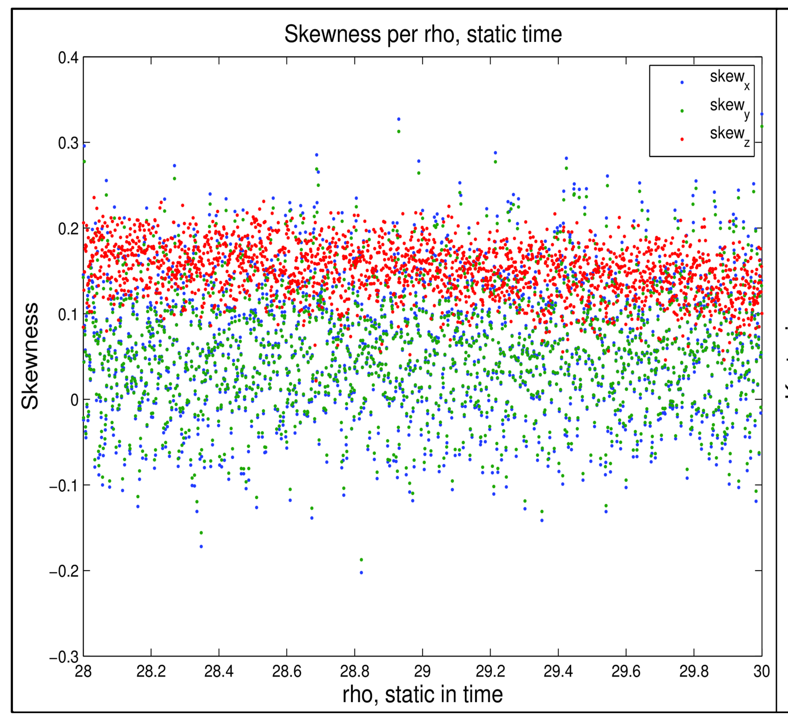


Fig 5: Skewnesses vs static ρ

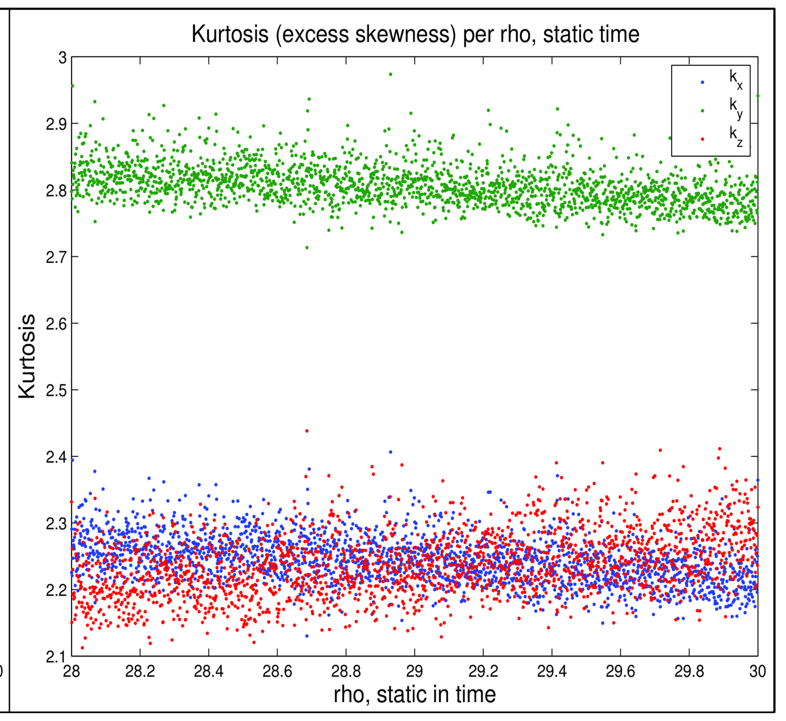


Fig 6: Kurtoses vs. static ρ

## Results: $\rho$ dynamic

We then collect a trajectory of 1500 time length (180k pts) with the parameter $\rho(t)$ dynamic in time according to the scheme in Figure 2. An xy projection of a select 30k pts in the trajectory during the dynamic portion of $\rho(t)$ is presented in Figure 7. Notice the trajectory is more irregular than in the static cases; this affects the overall shape of the attractor basin slightly without changing topologically. We then calculated the same statistics and measures of shape for each 10k pt segment of this trajectory with dynamic $\rho(t)$. Typical results are presented in Figures 8 through 11.

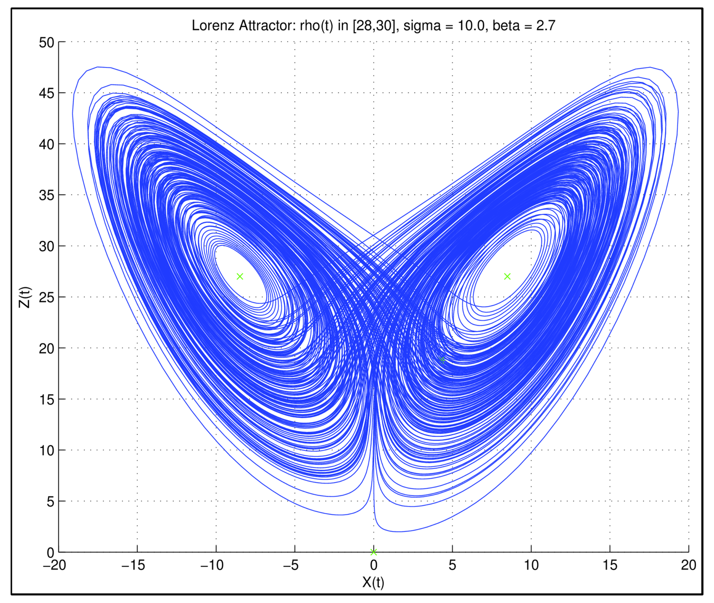


Fig 7: Trajectory with ρ = ρ(t)

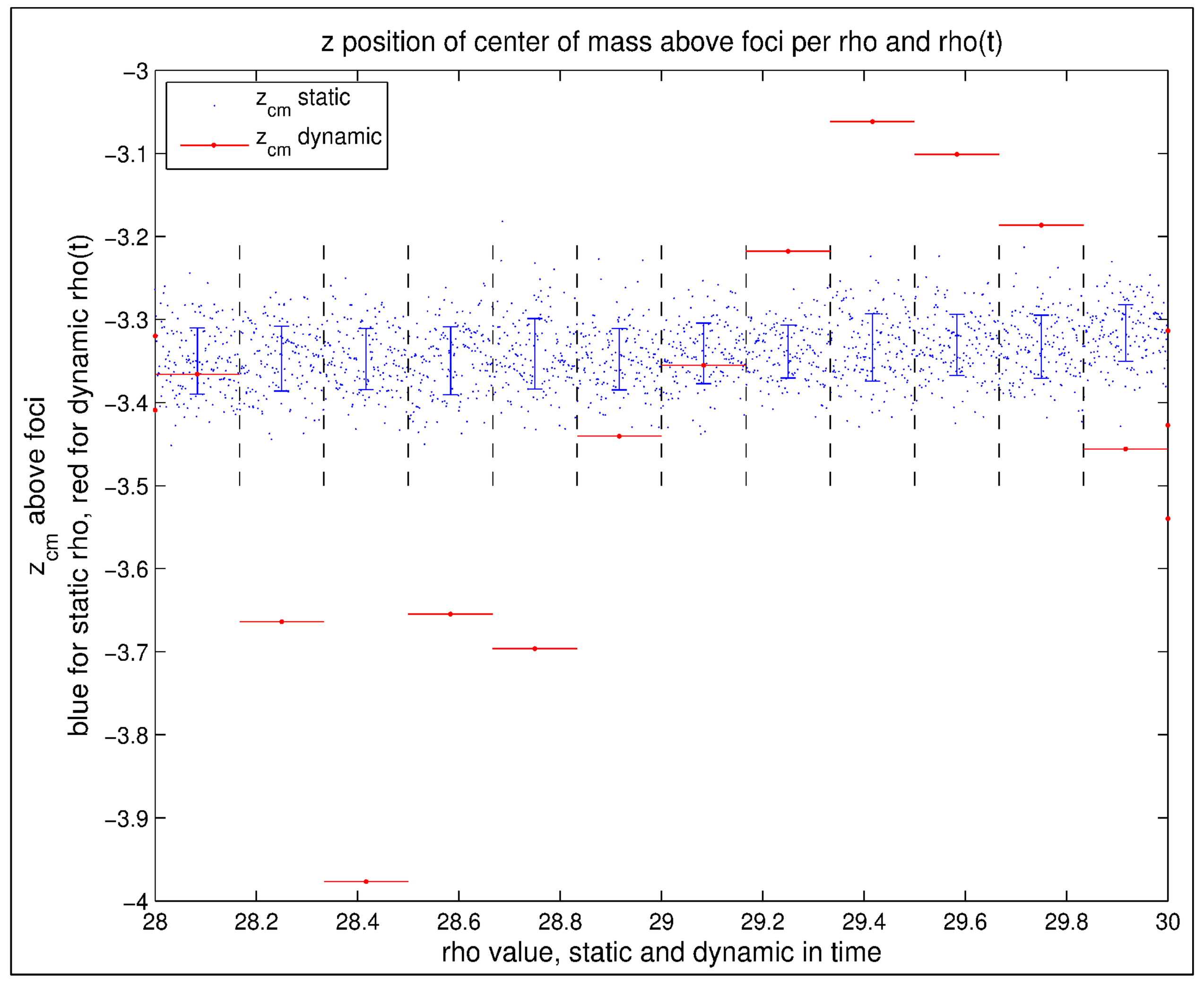


Fig 8: Height of $z_{cm}$, comparison

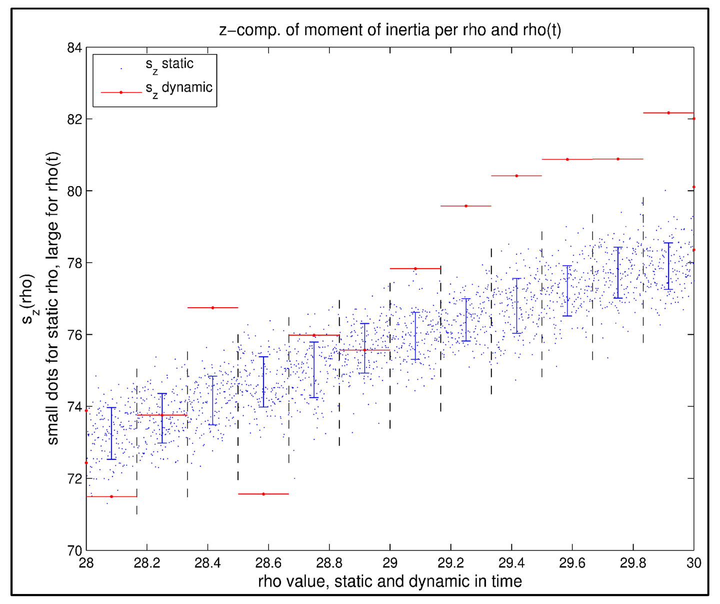


Fig 9: 2nd moment in z, comparison

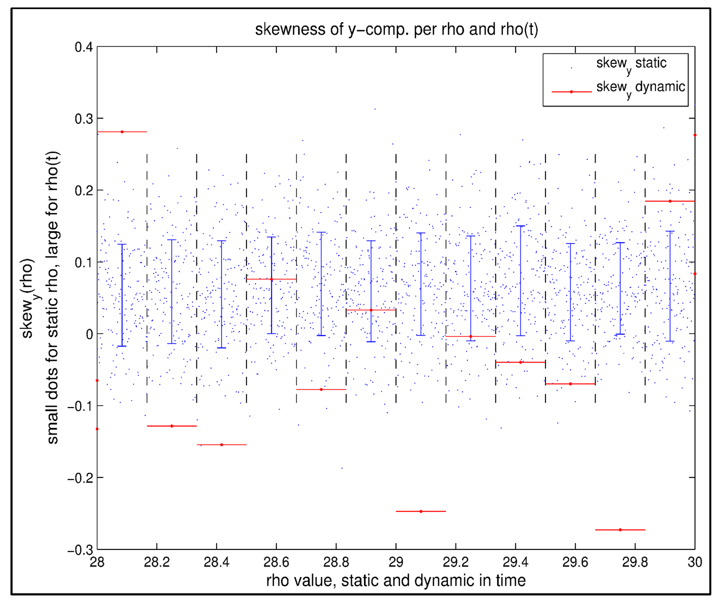


Fig 10: Skewness in y, comparison

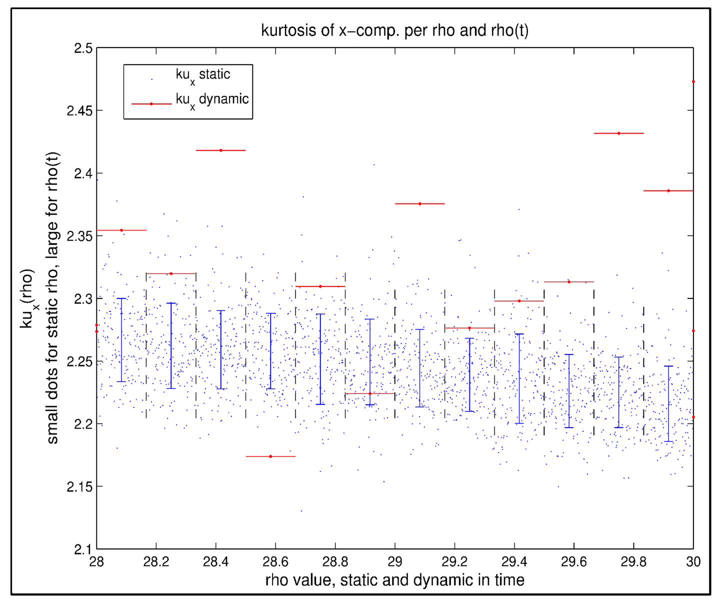


Fig 11: Kurtosis in x, comparison

Figures 1 through 8: Dashed black lines indicate where the $\rho$ space was segmented to calculate the statistics and measures of shape. Vertical blue bars indicate a range of one standard deviation on each side of the mean of the static $\rho$ data. Red dots are the results using 10k pt slices of the dynamic $\rho(t)$ trajectory with a horizontal bar to represent the range of $\rho$. Observe the statistics and measures of shape for the trajectory with parameter $\rho(t)$ changing in time often lie several standard deviations away from the expected value for a collection of trajectories with static parameter $\rho$ in the same range. Additionally, the dynamic $\rho(t)$ statistics and measures return to the expected value once $\rho$ has stabilized again.

## Conclusions:

The variations in the shape of the attractor basin with respect to the parameter $\rho$ are well understood when $\rho$ is a static parameter. Although we have changed the dynamic parameter $\rho(t)$ slowly, the shape of the attractor basin deviates significantly from that manifested over the immediate range of values of $\rho$. Such changes in shape are quantified in deviations in the statistics and measures of shape from the expected value over static parameter sets, yielding an even stranger attractor. The transition from one strange attractor to a second for this system has been shown to be characterized by significant excursions of the underlying statistics.

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